State dependent matrices and balanced energy functions for nonlinear systems

Scherpen, Jacquelien M.A.; Gray, W. Steven

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Jacquelien M. A. Scherpen

Delft University of Technology
Fac. of Inf. Techn. & Syst.
Department of Electrical Engineering
P.O. Box 5031, 2600 GA Delft
The Netherlands
J.M.A.Scherpen@et.tudelft.nl

W. Steven Gray

Dept. of Electrical and
Computer Engineering
Old Dominion University
Norfolk, Virginia 23529-0246
U.S.A.
gray@ece.odu.edu

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Abstract

The nonlinear extension of the balancing procedure requires the case of state dependent quadratic forms for the energy functions, i.e., the nonlinear extensions of the linear Gramians are state dependent matrices. These extensions have some interesting ambiguities that do not occur in the linear case. Namely, the choice of the state dependent matrix in the semi-quadratic form is not unique, and therefore may result in different eigenvalues. The introduction of so-called null-matrices is useful for the analysis of this problem. Furthermore, the concept of norm-preserving transformations provides further insight on these ambiguities. This paper provides a detailed analysis of this phenomenon and outlines some future directions for research.

1 Introduction

The notion of balanced realizations for nonlinear state space model reduction problems was first introduced by Scherpen in [12, 13]. Analogous to the Gramians matrices used in the linear case, e.g. [8], controllability and observability (energy) functions are used to determine how important each state component is in influencing the input-output map of the system. These functions are then transformed, through a change of coordinates, into a simultaneous diagonal form in order to identify the so called *singular value functions* of the system. In the linear case, these functions are equivalent to the square of the (constant) Hankel singular values of the system. State truncation is finally accomplished by examining the singular value functions in a neighborhood of 0 and deleting states that correspond to the smallest singular value functions in a local sense.

The procedure for nonlinear balancing, however, has some interesting ambiguities that do not occur in the linear case. Specifically, it appears that the singular value functions defined in [12, 13] are dependent on a particular factorization of the observability function which follows from the Fundamental Theorem of Integral Calculus. It has been shown in [3, 4] that in a fixed coordinate frame this factorization is not unique, and thus other distinct definitions for the singular value functions are possible. Of course, this is of great concern in model reduction applications, e.g., [9, 10], since decisions about state deletion should only depend on the coordinate frame of the state space and on intrinsic qualities of input-output map. In this paper we give a thorough analysis of this problem, consider what is required for a balancing procedure to be *consistent*, and study the role of norm preserving and orthogonal coordinate transformations.

The paper is organized as follows. In Section 2, the background for the problem is provided by reviewing some standard definitions in connection with nonlinear balanced realizations. Furthermore, the nature of our main problem is explained and illustrated with an example. In Section 3, we first consider the non-uniqueness of energy function factorizations via so called *null matrix functions*, as given in [3, 4]. Section 4 studies norm preserving coordinate transformations, and studies the effect of such transformations on singular value functions. Also, orthogonal coordinate transformations, which are a subclass of norm preserving transformations, are considered, and it is shown that they exhibit "singular value function" preserving properties. In Section 5 we present the requirements for a consistent factorization procedure for energy functions and balancing coordinate transformations. We conclude that the procedure which follows from the Fundamental Theorem of Integral Calculus does not fulfill these requirements. Finally, in Section 6 we summarize our conclusions.

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^{\mathrm{n}}$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2 \, dt}$. If $L : \mathbb{R}^{\mathrm{n}} \mapsto \mathbb{R}$ is a differen-

tiable function, then its partial derivative $\frac{\partial L}{\partial x}$ will be the row vector of partial derivatives $\frac{\partial L}{\partial x_i}$ where $i = 1, \ldots, n$.

2 The nature of the problem

In this section, the background for the problem is first briefly outlined by reviewing some standard definitions in connection with nonlinear balanced realizations. All of this material has been adapted from [12, 13]. Then a simple example is provided to illustrate the non-uniqueness phenomena considered in this paper.

Let $\mathcal{M}$ be an n-dimensional smooth manifold, and let

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

be a system defined in terms of local coordinates on $\mathcal{M}$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. We assume that f, g and h are smooth on $\mathcal{M}$, $f(0) = 0$ and $h(0) = 0$. The corresponding controllability and observability functions (or energy functions, collectively) for such a system are defined below.

Definition 2.1 *The* **controllability** *and* **observability functions** *for the system (f, g, h) are defined, respectively, as*

$$L_c(x) = \min_{\substack{u \in L_2(-\infty, 0) \\ x(-\infty)=0,\ x(0)=x}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 \, dt$$

and

$$L_o(x) = \frac{1}{2} \int_0^{\infty} \|y(t)\|^2 \, dt,$$

when $x(0) = x$, and $u(t) = 0$ for $0 \leq t < \infty$.

In order for a balanced realization to exist, the following properties of the system are assumed throughout the paper:

1. f is asymptotically stable on some neighborhood Y of 0.
2. The system (f, g, h) is zero-state observable on Y.
3. L_c and L_o exist and are smooth on Y.

The next collection of results form the core of the standard nonlinear balancing procedure. The lemma below is just a specialization of the Fundamental Theorem of Integral Calculus.

Lemma 2.1 [7] *Let L be a smooth real-valued function on a convex neighborhood $V \subset \mathbb{R}^n$ of 0 with L(0)=0. Then L exhibits the factorization*

$$L(x) = a^T(x)x,$$

where a is the smooth vector field on V with component functions

$$a_i(x) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1, \ldots, tx_n) \, dt.$$

Observe that $a^T(0) = \frac{\partial L}{\partial x}(0)$, and in fact *any* factorization of the form $L(x) = \tilde{a}^T(x)x$ necessarily has the property that $\tilde{a}^T(0) = \frac{\partial L}{\partial x}(0)$. The following lemma comes from applying Morse's Lemma to L_c [7], and the above lemma twice to L_o.

Lemma 2.2 *For a system (f, g, h) with corresponding energy functions (L_c, L_o), there exists a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$, defined on a neighborhood V of 0 which converts the system into an* **input-normal realization**, *where*

$$\begin{aligned} \bar{L}_c(\bar{x}) &:= L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x} \\ \bar{L}_o(\bar{x}) &:= L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} \end{aligned}$$

with M an $n \times n$ symmetric matrix-valued function having smooth component functions on $\bar{V} := \phi^{-1}(V)$ and $M(0) = \frac{\partial^2 L_o}{\partial x^2}(0)$.

Analogous to the above observation, *any* factorization of the form $\bar{L}_o(\bar{x}) = \frac{1}{2}\bar{x}^T M'(\bar{x})\bar{x}$ necessarily has the property that $M'(0) = \frac{\partial^2 L}{\partial x^2}(0)$. In order to diagonalize M, the following technical lemma is needed.

Lemma 2.3 [5] *If there exists a neighborhood $\bar{V}$ of 0, where the number of distinct eigenvalues of M is constant everywhere $\bar{V}$, then the eigenvalues and orthonormalized eigenvectors (λ_i, p_i), $i = 1, \ldots, n$ of M are smooth functions of $\bar{x} \in \bar{V}$.*

Theorem 2.1 *For a system (f, g, h) satisfying the condition in Lemma 2.3, there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, defined on a neighborhood U of 0 which converts the system into a* **input-normal/output-diagonal realization**, *where*

$$\begin{aligned} \tilde{L}_c(z) &:= L_c(\psi(z)) = \frac{1}{2}z^T z, \\ \tilde{L}_o(z) &:= L_o(\psi(z)) = \frac{1}{2}z^T diag(\tau_1(z), \ldots, \tau_n(z))z \end{aligned}$$

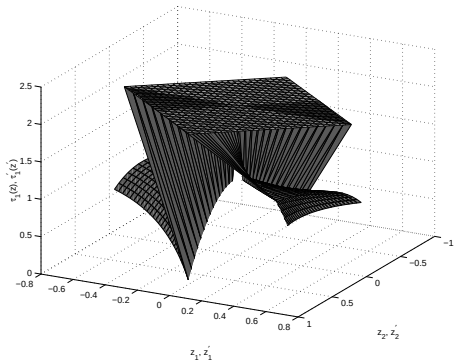
with $\tau_1(z) \geq \ldots \geq \tau_n(z)$ being smooth functions on $W := \psi^{-1}(U)$.

The set of functions τ_i, $i = 1, \ldots, n$ are called the *singular value functions* of (f, g, h). The final step of this balancing procedure is given below.

Theorem 2.2 *For the system in Theorem 2.1, there exists a coordinate transformation $z = \eta(\bar{z})$, $\eta(0) = 0$, defined on the neighborhood W of 0 which converts the system into a* **balanced realization**, *where*

$$\begin{aligned} \breve{L}_c(\bar{z}) &:= \tilde{L}_c(\eta(\bar{z})) \\ &= \frac{1}{2}\bar{z}^T diag(\sigma(\bar{z}_1)^{-1}, \ldots, \sigma(\bar{z}_n)^{-1})\bar{z} \\ \breve{L}_o(\bar{z}) &:= \tilde{L}_o(\eta(\bar{z})) \\ &= \frac{1}{2}\bar{z}^T diag(\tilde{\sigma}_1(\bar{z}), \ldots, \tilde{sigma}_n(\bar{z}))\bar{z}, \end{aligned}$$

with $\tilde{\sigma}_i(z) = \sigma_i(\bar{z}_i)^{-1}\tau_i(\eta^{-1}(\bar{z}))$ and $\sigma_i(\bar{z}_i) := \tau_i(0, \ldots, 0, \eta_i^{-1}(\bar{z}_i), 0, \ldots, 0)^{\frac{1}{2}}$ for $i = 1, \ldots, n$.



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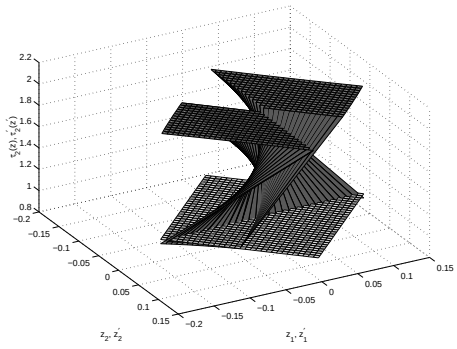
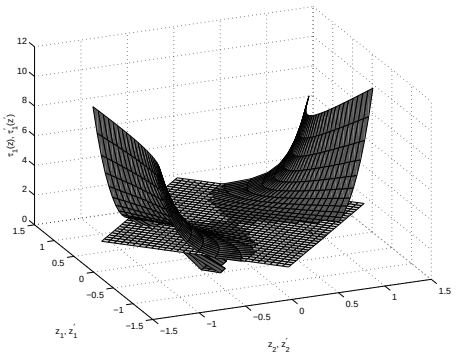


(b)

Figure 1: The singular value functions for Example 2.1 when $c_1(x) = c_2(x) = 0$ (light gray), and when $c_1(x) = x_1$ and $c_2(x) = x_2$ (dark gray).



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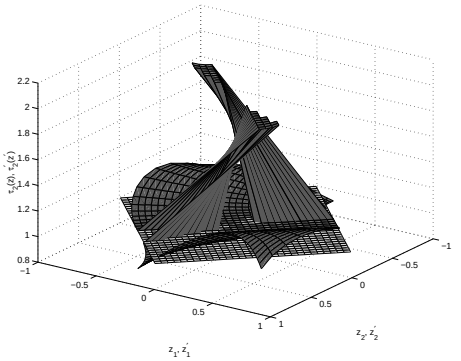

(b)

Figure 2: The singular value functions for Example 2.1 when $c_1(x) = c_2(x) = 0$ (light gray), and when $c_1(x) = x_2^3$ and $c_2(x) = -3x_1^3$ (dark gray).

Note that along coordinate axes it is easily verified for $i = 1, \ldots, n$ that:

$$\begin{array}{rcl} \breve{L}_c(0,\ldots,0,\bar{z}_i,0,\ldots,0) & = & \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1} \\ \breve{L}_o(0,\ldots,0,\bar{z}_i,0,\ldots,0) & = & \frac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i). \end{array} \qquad (1)$$

To illustrate the non-uniqueness features in the above balancing procedure, consider the following example.

Example 2.1 Consider the system well defined on an open neighborhood in $\mathbb{R}^2$:

$$\begin{array}{rcl} f(x) & = & -\begin{bmatrix} \alpha^2 x_1 + 2\alpha x_2 + (\alpha^2-2)x_2^2 \\ x_2 \end{bmatrix} \\ g(x) & = & \sqrt{2}\begin{bmatrix} \alpha - 2x_2 \\ 1 \end{bmatrix} \\ h(x) & = & \frac{1}{\sqrt{3}}(3\alpha(x_1+x_2^2) + (\alpha - 2\sqrt{2})x_2), \end{array}$$

where $\alpha = (\sqrt{3}+\sqrt{2})(\sqrt{3}+2)$. The corresponding energy functions can be shown to be

$$\begin{array}{rcl} L_c(x) & = & \frac{1}{2}(x_1^2 + 2x_1x_2^2 + x_2^2 + x_2^4) \\ L_o(x) & = & \frac{1}{4}(3x_1^2 + 2x_1x_2 + 6x_1x_2^2 + 3x_2^2 + 2x_2^3 + 3x_2^4) \end{array}$$

for all $x \in \mathbb{R}^2$. Now applying the coordinate transformation

$$x = \phi(\bar{x}) = \begin{bmatrix} \bar{x}_1 + \bar{x}_1^2 \\ \bar{x}_2 \end{bmatrix}$$

yields an input-normal form with energy functions:

$$\begin{array}{l} L_c(\bar{x}) = \frac{1}{2}\,\bar{x}^T\bar{x} \\ L_o(\bar{x}) = \frac{1}{2}\bar{x}^T M(\bar{x})\bar{x} = \frac{1}{2}\,\bar{x}^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \bar{x}. \end{array}$$

Since M is constant in this representation, the singular value functions appear to be the constant functions: $\tau_1(z) = 2,\ \tau_2(z) = 1$ in the diagonalized coordinate frame $x = \psi(z)$. The situation is, however, more complex than it first appears. For example, consider the smooth symmetric matrix function

$$A(\bar{x}) = c_1(\bar{x})\begin{bmatrix} -2\bar{x}_2 & \bar{x}_1 \\ \bar{x}_1 & 0 \end{bmatrix} + c_2(\bar{x})\begin{bmatrix} 0 & \bar{x}_2 \\ \bar{x}_2 & -2\bar{x}_1 \end{bmatrix},$$

where $c_1, c_2 \in C^\infty(\mathbb{R}^2)$, the ring of smooth real-valued functions defined on $\mathbb{R}^2$. Since $\bar{x}^T A(\bar{x})\bar{x} = 0$ everywhere on $\mathbb{R}^2$ and $A(0) = 0$, another input-normal form in the *same*

coordinate system is:

$$\begin{aligned}\bar{L}_c(\bar{x}) &= \tfrac{1}{2}\,\bar{x}^T\bar{x}\\ \bar{L}_o(\bar{x}) &= \tfrac{1}{2}\bar{x}^T(M(\bar{x})+A(\bar{x}))\bar{x}\\ &:= \tfrac{1}{2}\bar{x}^T M'(\bar{x})\bar{x}\\ &= \tfrac{1}{2}\,\bar{x}^T\begin{bmatrix} m'_{11}(\bar{x}) & m'_{12}(\bar{x})\\ m'_{12}(\bar{x}) & m'_{22}(\bar{x})\end{bmatrix}\bar{x}.\end{aligned} \tag{2}$$

where $m'_{11}(\bar{x}) = \frac{3}{2} - 2c_1(\bar{x})\bar{x}_2$, $m'_{12}(\bar{x}) = \frac{1}{2} + c_1(\bar{x})\bar{x}_1 + c_2(\bar{x})\bar{x}_2$, and $m'_{22}(\bar{x}) = \frac{3}{2} - 2c_2(\bar{x})\bar{x}_1$. For most choices of c_1, c_2, the condition in Lemma 2.3 is satisfied, and thus M' is smoothly diagonalizable. Consider, for example, the case: $c_1(\bar{x}) = \bar{x}_1$ and $c_2(\bar{x}) = \bar{x}_2$. Then it follows that the eigenvalues of M' are $\lambda'_1(\bar{x}) = 2 + (\bar{x}_1 - \bar{x}_2)^2$ and $\lambda'_2(\bar{x}) = 1 - (\bar{x}_1 + \bar{x}_2)^2$, which are distinct everywhere on $\mathbb{R}^2$. The diagonalizing transformation

$$x = \psi'(z') = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{2}z'_1 - z'_2 & \frac{1}{\sqrt{2}} + \frac{1}{2}z'_2\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{bmatrix} z'$$

yields the corresponding input-normal/output-diagonal form:

$$\begin{aligned}\tilde{L}'_c(z') &:= L_c(\psi'(z')) = \frac{1}{2}(z')^T z',\\ \tilde{L}'_o(z') &:= L_o(\psi'(z'))\\ &= \frac{1}{2}(z')^T diag(\tau'_1(z'), \tau'_2(z'))z'\\ &= \frac{1}{2}(z')^T diag(2 + 2(z'_2)^2, 1 - 2(z'_1)^2)z'.\end{aligned}$$

Thus, it is clear that a different factorization of L_o, via the introduction of the matrix-valued function A, leads to a different set of singular value functions. Note, however, that they *are* identical along respective coordinate directions, i.e., $\tau'_i(0,\ldots,0,z'_i,0,\ldots,0) = \tau_i(0,\ldots,0,z_i,0,\ldots,0)$ for i=1,2. This is also illustrated in Figure 1. However, notice in Figure 2 that this result does not hold for every set of c_i functions. Furthermore, observe that any coordinate transformation of the form $x = \nu(w) = T(w)w$ with $T^T(w)T(w) = I$ transforms the energy functions in (2) to yet another input-normal/output-diagonal form after applying the diagonalizing transformation $w = \hat{\psi}(y)$:

$$\begin{aligned}\hat{L}_c(y) &:= L_c((\nu\circ\hat{\psi})(y)) = \frac{1}{2}y^T y,\\ \hat{L}_o(y) &:= L_o((\nu\circ\hat{\psi})(y)) = \frac{1}{2}y^T diag(\hat{\tau}_1(y), \hat{\tau}_2(y))y,\end{aligned}$$

where $\hat{\tau}_i(y) = \lambda_i((\nu\circ\hat{\psi})(y))$, $i = 1, 2$. Thus seemingly different sets of singular value functions are potentially related by an orthogonal coordinate transformation, but that is not readily apparent in this example. In the next section these issues are considered in detail.

3 A source of non-uniqueness

In this section a source of non-uniqueness in computing the singular value functions of a system is presented: the addition of a null matrix function.

Let V be an open neighborhood of 0, and let $C^\infty(V)$ denote the Abelian ring of smooth real-valued functions defined on V. (Addition and multiplication are defined in the obvious pointwise fashion on V, see for example [6].) Let $M_n(C^\infty(V))$ denote the set of $n \times n$ matrices with components from $C^\infty(V)$. Using the usual notions of matrix addition and multiplication, $M_n(C^\infty(V))$ is an associative ring with identity [2]. The subset $S_n(C^\infty(V))$ consists of all symmetric matrices in $M_n(C^\infty(V))$. The following subset of $S_n(C^\infty(V))$ is most relevant in this paper.

Definition 3.1 *The subset $\mathcal{A}(V) \subset S_n(C^\infty(V))$ is the set of matrix-valued functions, A, with the following properties:*

i. $A(0) = 0$.

ii. $x^T A(x)x = 0, \forall x \in V$.

Any $A \in \mathcal{A}(V)$ is called a **null matrix function** on V. Some properties of $\mathcal{A}(V)$ are considered in the following lemma, and then an application of this idea is given in the subsequent lemma.

Lemma 3.1 *For any neighborhood V of 0, the following statements are true:*

i. *$\mathcal{A}(V)$ is a vector space over R.*

ii. *$\mathcal{A}(V)$ is a module over $C^\infty(V)$.*

iii. *The matrix $A \equiv 0$ is the only constant matrix in $\mathcal{A}(V)$.*

iv. *The relation $M \sim M' \Leftrightarrow M - M' \in \mathcal{A}(V)$ is an equivalence relation on $S_n(C^\infty(V))$.*

Proof: Proofs of these statements are elementary. ∎

Lemma 3.2 *On any neighborhood V of 0 and for any $M, M' \in S_n(C^\infty(V))$*

$$x^T M(x)x = x^T M'(x)x,\ x \in V \Leftrightarrow M \sim M'.$$

Proof: The proof is trivial using the fact that the equivalence on the left-hand side also implies $M(0) = M'(0)$ ∎

An interesting observation about the set $\mathcal{A}(V)$ is its relationship to an isotropy subgroup of the matrix group:

$$\begin{aligned}&GL_n(C^\infty(V)) :=\\ &\{T \in M_n(C^\infty(V)) : \exists S \in M_n(C^\infty(V)) \text{ with } TS = I\},\end{aligned}$$

where I denotes the identity matrix [11]. Viewing $GL_n(C^\infty(V))$ as a transformation group on V with the usual group action

$$\begin{aligned}\psi &: GL_n(C^\infty(V)) \times V \mapsto V\\ &: (T, x) \mapsto T(x)x,\end{aligned}$$

the isotropy subgroup for any $x \in V$ is

$$I_x := \{T \in GL_n(C^\infty(V)) : T(x)x = x\}.$$

The corresponding isotropy subgroup for V is

$$I_V := \bigcap_{x \in V} I_x.$$

Now given any symmetric element $B \in I_V$, it is immediate that $I - B \in \mathcal{A}(V)$, that is,

$$x^T(I - B(x))x = x^T(x - B(x)x) = 0.$$

However, it is easy to find examples of null matrices with no corresponding element in I_V. Specifically, it is possible for $x^T A(x)x = 0$ everywhere on V without $A(x)x = 0$. Hence, the usual methods associated with matrix groups do not completely describe the nature of $\mathcal{A}(V)$.

Returning now to the main problem, it was observed in the example from the previous section that the equivalence $M \sim M'$ on $S_n(C^\infty(V))$ does not imply equivalence of their respective pointwise spectra. This is a fundamental source of non-uniqueness in the calculation of the singular value functions of a system. However, it is still possible to make some general statements relating their spectra. This is done using the following results, from which the proofs can be found in [3, 4].

Lemma 3.3 *If $A \in \mathcal{A}(V)$ then $A(x) = [a_{ij}(x)] = [\alpha_{ij}(x)x] = [\sum_{k=1}^n (\alpha_{ij}(x))_k x_k] := [\sum_{k=1}^n \alpha_{ijk}(x)x_k]$ on V, where*

i. $\alpha_{ijk}(0) = \frac{\partial a_{ij}}{\partial x_k}(0)$*;*

ii. $\alpha_{ijk}(0) + \alpha_{kij}(0) + \alpha_{jki}(0) = 0$ *for all i, j, k;*

iii. $\sum_{ijk}(\alpha_{ijk}(x) + \alpha_{kij}(x) + \alpha_{jki}(x))x_i x_j x_k = 0$ *on V.*

Next consider the following result from matrix perturbation theory adapted from [1] (see p. 163).

Theorem 3.1 *Let $M_0 \in \mathbb{R}^{n \times n}$ be a simple symmetric matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i\}_{i=1}^n$. For $\theta \in \mathbb{R}$ and symmetric matrices $M_1, M_2 \in \mathbb{R}^{n \times n}$ define*

$$M(\theta) = M_0 + M_1\theta + M_2\theta^2.$$

For sufficiently small $|\theta|$, the matrix $M(\theta)$ is also simple, and its corresponding eigenvalues $\{\lambda_i(\theta)\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i(\theta)\}_{i=1}^n$ depend analytically on θ, i.e.,

$$\begin{aligned} \lambda_i(\theta) &= \lambda_i^{(0)} + \lambda_i^{(1)}\theta + \lambda_i^{(2)}\theta^2 + \dots \\ p_i(\theta) &= p_i^{(0)} + p_i^{(1)}\theta + p_i^{(2)}\lambda^{(2)}\theta^2 + \dots \end{aligned}$$

for $i = 1, 2, \dots, n$. In particular,

$$\begin{aligned} \lambda_i^{(0)} &= \lambda_i \\ \lambda_i^{(1)} &= p_i^T M_1 p_i \\ \lambda_i^{(2)} &= p_i^T M_2 p_i + \sum_{\substack{j=1 \\ i \neq j}}^N \frac{1}{\lambda_i - \lambda_j} |p_i^T M_1 p_j|^2 \\ p_i^{(0)} &= p_i \\ p_i^{(1)} &= \sum_{\substack{j=1 \\ i \neq j}}^N \frac{p_i^T M_1 p_j}{\lambda_i - \lambda_j}. \end{aligned}$$

Now we can relate the singular value functions on the coordinate axes of different factorizations as follows [3, 4].

Theorem 3.2 *Suppose $M \in S_n(C^\infty(V))$ and $M(0)$ is simple. Let $\{\lambda_i, p_i\}$ denote the smoothly defined eigenvalue and orthonormal eigenvector pairs for M on a neighborhood $\bar{V} \subset V$ of 0 (c.f. Lemma 2.3). Let $A \in \mathcal{A}(V)$ and define $M' = M + A$ with corresponding eigenvalues $\{\lambda_i'\}_{i=1}^n$. In the diagonalized coordinate frame $z = \psi^{-1}(x)$ for M, the eigenvalues of M and M' are equivalent to first order along their respective coordinate directions. That is, sufficiently close to 0*

$$\begin{aligned} &\lambda_i'(\psi(0, \dots, 0, z_i, 0, \dots, 0)) = \\ &\lambda_i(\psi(0, \dots, 0, z_i, 0, \dots, 0)) + \mathcal{O}(z_i^2). \end{aligned} \tag{3}$$

Remarks:

1. In the context of the singular value functions, i.e., when $L_o(x) = \frac{1}{2}x^T M(x)x$ and $L_o'(x) = \frac{1}{2}x^T M'(x)x$, the identity (3) becomes

   $$\begin{aligned} &\lambda_i'(\psi(0, \dots, 0, z_i, 0, \dots, 0)) = \\ &\tau_i(0, \dots, 0, z_i, 0, \dots, 0) + \mathcal{O}(z_i^2). \end{aligned}$$

   The lefthand side of this identity is only equivalent to the true singular value functions for M' if the (orthogonal) diagonalizing transformation $z' = (\psi')^{-1}(x)$ for M' is identical to the diagonalizing transformation $z = \psi^{-1}(x)$ for M. This is the case in Example 2.1 from the previous section, M and M' are simultaneously diagonalized by the same coordinate transformation. In Figure 3, the theorem is illustrated for this example using various sets of c_i functions. The effect of an orthogonal coordinate transformation on singular value functions is considered in the next section.

2. In general the identity (3) is not true to second order or higher. However, if matrix $B_1 = 0$ in the proof of Theorem 3.2 then equality up to second order follows from the expression for $\lambda_i^{(2)}$ in Theorem 3.1. This is exactly the case in Example 2.1 for the first choice of functions c_1 and c_2.

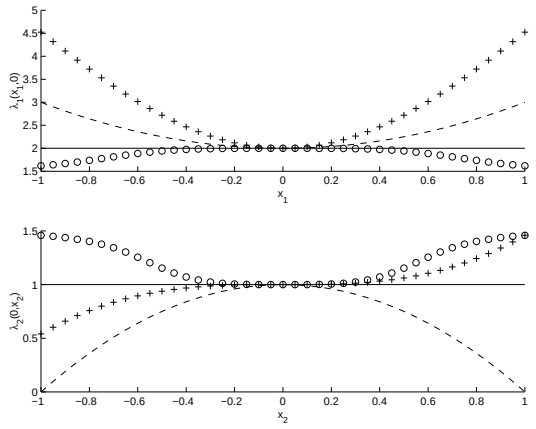

Figure 3: Coordinate axis cross sections of the functions λ_1 and λ_2 in Example 2.1 when $c_1(x) = c_2(x) = 0$ (solid line), $c_1(x) = x_1$ and $c_2(x) = x_2$ (dashed line), $c_1(x) = x_2^3$ and $c_2(x) = -3x_1^3$ (marked 'o'), and $c_1(x) = \cos(x_1) - 1$ and $c_2(x) = 3\sin(x_2)$ (marked '+').

4 Norm preserving coordinate transformations

A smooth coordinate transformation $x = \nu(w)$ is said to be *norm preserving* on a convex neighborhood of the origin, W, if $\|x\| = \|w\|$ for all $w \in W$. Since all such maps satisfy $\nu(0) = 0$, it follows directly from Lemma 2.1 that there exists at least one factorization of the form $\nu(w) = T(w)w$ where $T \in M_n(C^\infty(W))$. Thus, it is immediate that everywhere on W

$$\|\nu(w)\|^2 = w^T T^T(w)T(w)w = w^T w$$

or equivalently $T^T(w)T(w) = I + A_T(w)$ for some $A_T \in \mathcal{A}(W)$. In the context of energy functions, norm preserving transformations are interesting because they preserve input-normal forms, that is,

$$L_c(x) = \frac{1}{2}x^T x = \frac{1}{2}w^T w = \hat{L}_c(w).$$

A specific class of norm preserving transformations are the so called *orthogonal* transformations, which are characterized by having a factorization $\nu(w) = T(w)w$ where $T^T(w)T(w) = I$ for all $w \in W$. In the following theorem, it is observed that orthogonal coordinate transformations also preserve the singular value functions in a natural sense.

Theorem 4.1 *Consider a system (f, g, h) with singular value functions τ_i, $i = 1, \ldots, n$ derived from a specific input-normal form: $L_c(x) = \frac{1}{2}x^T x$, $L_o(x) = \frac{1}{2}x^T M(x)x$. Any orthogonal coordinate transformation, $x = \nu(w) = T(w)w$, yields the corresponding singular value functions*

$$\hat{\tau}_i = \tau_i \circ \psi^{-1} \circ \nu \circ \hat{\psi}, \quad i = 1, \ldots, n, \tag{4}$$

where $x = \psi(z)$ and $w = \hat{\psi}(y)$ are diagonalizing transformations for $M(\cdot)$ and $M(\nu(\cdot))$, respectively.

Proof : After applying the coordinate transformation ν and using the orthogonality condition, the new system has the input-normal form

$$\hat{L}_o(w) = \frac{1}{2}w^T \underbrace{T^{-1}(w)M(\nu(w))T(w)}_{\hat{M}(w)} w. \tag{5}$$

Hence, it follows that the matrices $M(\nu(w))$ and $\hat{M}(w)$ have the same eigenvalues for each w. To compute the singular value functions starting from $\hat{M}(\cdot)$, use the fact that $x = \psi(z) = T_\psi(z)z$ diagonalizes $M(x)$ in the appropriate fashion, i.e.,

$$T_\psi^T(z)M(\psi(z))T_\psi(z) = diag(\tau_1(z), \ldots, \tau_n(z)).$$

Consequently,

$$\begin{aligned}
&T_\psi^T(\psi^{-1}(\nu(w))) \cdot T(w)\hat{M}(w)T^{-1}(w) \cdot T_\psi(\psi^{-1}(\nu(w))) \\
&= T_\psi^T(\psi^{-1}(\nu(w)))\ M(\psi(\psi^{-1}(\nu(w))))\ T_\psi(\psi^{-1}(\nu(w))) \\
&= diag(\tau_1(\psi^{-1} \circ \nu(w)), \ldots, \tau_n(\psi^{-1} \circ \nu(w))) \\
&= diag(\hat{\tau}_1(y), \ldots, \hat{\tau}_n(y)),
\end{aligned}$$

where $y = \hat{\psi}^{-1}(w)$ is the diagonalized coordinate frame for $\hat{M}(w)$. Equating the diagonal terms on the righthand side of the last two equations gives

$$\hat{\tau}_i(y) = \tau_i(\psi^{-1} \circ \nu(\hat{\psi}(y))).$$

Hence, the theorem is proven. ∎

In Example 2.1, the first set of singular value functions were the constant values $\tau_1(z) = 2$ and $\tau(z) = 1$. Thus, in light of equation (4), they are invariant under all orthogonal coordinate transformations. This eliminates the possibility that such a transformation can relate $\{\tau_i\}$ to the second set of singular value transformations $\{\tau_i'\}$ derived from adding a null matrix. The next theorem, describes the effect of a general norm preserving coordinate transformation on a given set of singular value functions. This result is a combination of Theorems 3.2 and 4.1 and provides more possibilities to relate the various non-uniqueness phenomena.

Theorem 4.2 *Consider a system (f, g, h) with singular value functions τ_i, $i = 1, \ldots, n$ derived from a specific input-normal form: $L_c(x) = \frac{1}{2}x^T x$, $L_o(x) = \frac{1}{2}x^T M(x)x$ defined on a neighborhood V of 0 with $M \in S_n(C^\infty(V))$ and $M(0)$ simple. Any norm preserving coordinate transformation, $x = \nu(w) = T(w)w$, yields the following singular value functions expressed in the diagonalized coordinate frame for $M(\nu(\cdot))$ functions*

$$\begin{aligned}
&\lambda_i'(\hat{\psi}(0, \ldots, 0, y_i, 0, \ldots, 0)) = \\
&(\tau_i \circ \psi^{-1} \circ \nu \circ \hat{\psi})(0, \ldots, 0, y_i, 0, \ldots, 0) + \mathcal{O}(y_i^2), \\
&i = 1, \ldots, n,
\end{aligned}$$

for y_i sufficiently close to 0, and where $x = \psi(z)$ and $w = \hat{\psi}(y)$ are the diagonalizing transformation for $M(\cdot)$ and $M(\nu(\cdot))$, respectively.

Proof : Applying the coordinate transformation ν and using the identity

$$T^T(w) = T^{-1}(w)[I + T(w)A_T(w)T^{-1}(w)],$$

where A_T is a null matrix, gives a new input-normal form where

$$\check{L}_o(w) = \frac{1}{2}w^T\{[I + A_T(w)] \underbrace{T^{-1}(w)M(\nu(w))T(w)}_{\hat{M}(w)}\}w$$

(c.f. equation (5)). Letting

$$M'(w) = \hat{M}(w) + A_T(w)\hat{M}(w),$$

the proof proceeds similar to that of Theorem 4.1, using

$$\begin{aligned} M'(x) &= M(x) + A(x) \\ &= P(x)\Lambda(x)P^T(x) + A(x) \\ \underbrace{P^T(x)M'(x)P(x)}_{N(x)} &= \Lambda(x) + \underbrace{P^T(x)A(x)P(x)}_{B(x)}. \end{aligned} \tag{6}$$

That is, let $\hat{M} = P\Lambda P^T$ be the spectral decomposition near the origin. Then

$$\underbrace{P^T(w)M'(w)P(w)}_{N(w)} = \Lambda(w) + \underbrace{P^T(w)A_T(w)P(w)}_{B(w)}\Lambda(w).$$

After setting $y = P^T T^{-1} w = \hat{\psi}^{-1}(w)$ it follows that

$$\begin{aligned} N(\hat{\psi})(y) &= \Lambda(\hat{\psi(y)})B(\hat{\psi}(y))\Lambda(\hat{\psi}(y)) \\ \tilde{N}(y) &= \tilde{\Lambda}(y) + \tilde{B}(y)\tilde{\Lambda}(y). \end{aligned}$$

As before, $\tilde{N}$ has the same eigenvalues as M', and $\tilde{B}$ is a null matrix. Along the i-th coordinate direction and sufficiently close to the origin

$$\begin{aligned} &\tilde{N}(0,\ldots,0,y_i,0,\ldots,0) = \tilde{\Lambda}(0,\ldots,0,y_i,0,\ldots,0)+ \\ &\tfrac{d}{dy_i}\left[\tilde{B}(0,\ldots,y_i,\ldots,0)\tilde{\Lambda}(0,\ldots,y_i,\ldots,0)\right]\Big|_{y_i=0} y_i \\ &+\mathcal{O}(y_i^2) \\ &= \tilde{\Lambda}(0,\ldots,0,y_i,0,\ldots,0)+ \\ &\underbrace{\frac{d}{dy_i}\left[\tilde{B}(0,\ldots,0,y_i,0,\ldots,0)\right]\Big|_{y_i=0}}_{B_i}\tilde{\Lambda}(0)y_i + \mathcal{O}(y_i^2) \end{aligned}$$

Applying Lemma 3.3 and Theorems 3.1 and 4.1,

$$\begin{aligned} &\lambda_i'(\hat{\psi}(0,\ldots,0,y_i,0,\ldots,0)) = \\ &\hat{\lambda}(\hat{\psi}(0,\ldots,0,y_i,0,\ldots,0)) + e_i^T B_i\tilde{\Lambda}(0)e_i + \mathcal{O}(y_i^2) \\ &= (\tau_i \circ \psi^{-1} \circ \nu \circ \hat{\psi})(0,\ldots,0,y_i,0,\ldots,0)+ \\ &\quad e_i^T B_i e_i\ \lambda_i(0) + \mathcal{O}(y_i^2) \\ &= (\tau_i \circ \psi^{-1} \circ \nu \circ \hat{\psi})(0,\ldots,0,y_i,0,\ldots,0) + \mathcal{O}(y_i^2). \end{aligned}$$

This proves the theorem. ∎

The orthogonal transformations that are considered in Theorem 4.1 are a special case of the norm preserving transformations that are considered in Theorem 4.2. In general we prefer to work with orthogonal transformations due to their "eigenvalue" preserving properties. The following theorem gives conditions under which an orthogonal transformation can be subtracted from a given norm preserving transformation.

Theorem 4.3 *Suppose that $v(x) = V(x)x$ is a smooth, non-singular, norm preserving coordinate transformation on an open neighborhood W of 0, i.e., $v(x)^T v(x) = x^T x$, and $V(x)$ is a smooth $n \times n$ matrix. Then $A(x) := V(x)^T V(x) - I \in \mathcal{A}(W)$. Define a smooth $n \times n$ matrix $\Delta(x)$ such that $\Delta(x)x = 0$ for all $x \in W$. Then $v(x) = (V(x) + \Delta(x))x$ is an orthogonal transformation if $\Delta(x)$ fulfills the following state dependent matrix equation*

$$\Delta(x)^T V(x) + V(x)^T\Delta(x) + \Delta(x)^T\Delta(x) + A(x) = 0. \tag{7}$$

Proof : By definition $(V(x) + \Delta(x))x$ is an orthogonal factorization of $v(x)$ if

$$(V(x) + \Delta(x))^T(V(x) + \Delta(x)) = I.$$

Rewriting the latter equation, and using the expression for $A(x)$ yields (7). ∎

5 A coordinate free factorization procedure

In this section we formulate a problem that involves the search for a kind of "canonical" factorization procedure that exhibits some kind of consistency under arbitrary non-singular coordinate transformations. In particular, if two systems that are related by a non-singular coordinate transformation are brought into balanced form by the same procedure, then one would like to obtain diagonal forms that do not differ by a null matrix, i.e., their singular value functions will be identical.

Problem: Consider the smooth realization $(f(x), g(x), h(x))$ with a smooth energy function $L(x)$, which is related via a smooth non-singular coordinate transformation $z = \nu(x)$, $\nu(0) = 0$, to the realization $\tilde{f}(z), \tilde{g}(z), \tilde{h}(z))$ with energy function $\tilde{L}(z) = L(\nu^{-1}(z))$. Suppose that the energy function fulfills $L(0) = 0$ and $\frac{\partial L}{\partial x}(0) = 0$. It is well known that on a convex neighborhood of 0 there exist $n \times n$ matrices $V(x)$, $M(x)$, and $\tilde{M}(z)$ such that

$$\begin{aligned} \nu(x) &= V(x)x \\ L(x) &= x^T M(x)x \\ \tilde{L}(z) &= z^T\tilde{M}(z)z, \end{aligned}$$

where the entries of $V(x)$, $M(x)$ and $\tilde{M}(z)$ are smooth functions of x, x and z, respectively, and where $M(x) = M(x)^T$, $\tilde{M}(z) = \tilde{M}(z)^T$. It follows directly that $L(x) = \nu(x)^T \tilde{M}(\nu(x))\nu(x)$. Furthermore, for all factorizations it is clear that $M(0) = V(0)^T \tilde{M}(0) V(0)$. The main question is whether there exist a systematic factorization procedure to produce $V(x)$, $M(x)$, and $\tilde{M}(z)$ such that

$$V(x)^T \tilde{M}(\nu(x))V(x) = M(x). \tag{8}$$

The problem description is visualized in the diagram below

$$\begin{array}{ccc} (\tilde{f}(z), \tilde{g}(z), \tilde{h}(z), \tilde{L}(z)) & \overset{\tilde{L}-\text{factorization}}{\longrightarrow} & \tilde{M}(z) \\ {\scriptstyle z=\nu(x)} \downarrow & & {\scriptstyle V(x)} \downarrow \\ (f(x), g(x), h(x), L(x)) & \overset{L-\text{factorization}}{\longrightarrow} & M(x) \end{array}$$

■

It is easily verified by example that a procedure based on Lemma 2.1, resulting from the Fundamental Theorem of Integral Calculus, does not exhibit the required property (8), except when restricted to linear systems. Furthermore, in order to yield the usual linear case, an additional property for the procedure is required. Namely,

Additional requirement: If the coordinate transformation $\nu(x)$ is linear, then the factorization procedure should produce a constant matrix, i.e., $V(x)$ is a constant matrix. If the energy function $L(x)$ is a true quadratic form, then the factorization procedure must result in a constant matrix, i.e., $M(x)$ is a constant matrix. ■

Observe that with this additional requirement, it follows immediately that for norm preserving transformations, the procedure will produce an orthogonal factorization. Namely, since $M(x) = I$ and $\tilde{M}(z) = I$, then from (8) the required procedure results in $V(x)^T V(x) = I$.

6 Conclusions

In this paper we presented a study of energy functions which are state dependent quadratic forms, i.e., quadratic forms with a state dependent matrix. The factorization into state dependent quadratic forms is not unique, and thus different state dependent matrices result in distinct eigenvalue functions. This has among others consequences for nonlinear balancing methods, the corresponding singular value functions and model reduction.

In order to characterize different factorizations, so-called null matrices were introduced. It was shown that at least locally, to first order, the singular value functions are unique. Furthermore, preserve the singular value functions exactly in a natural sense. However, in general, there is a need for a coordinate free factorization procedure, one that results in the same singular value functions, starting from any admissible state space system. The search for such a procedure is a topic of future research.

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